8 Relations

8.1 Relations and Their Properties

1. for sets A and B we define a <u>relation from A to B</u> to be a subset of $A \times B$

2. for example the graph of a function is a relation from the domain (A) to its range (B) (For a review of functions see outline of Section 2.3 below). Note that every function is a relation because it relates an element of A to an element of B. However not every relation a function because a relation may relate an element of A to more than just one element of B (or also because the empty relation is not a function).

3. a <u>relation on a set A</u> is a relation from A to A (i.e. subset of $A \times A$, where A could be finite or infinite)

4. for a finite set A (with $|A| = n$), there are $2^{|A \times A|} = 2^{n^2}$ possible relations on A, namely the elements of the power set of $A \times A$

5. a relation R defined on A is <u>reflexive</u> if $\forall a \in A$, then $(a, a) \in R, \forall a \in A$

6. a relation R defined on A is <u>symmetric</u> if for elements $a, b \in A$ with $(a, b) \in R$, then $(b, a) \in R$

7. a relation R defined on A is <u>antisymmetric</u> if for elements $a, b \in A$ with $(a, b) \in R$ and $(b, a) \in R$, then $a = b$

8. a relation is <u>transitive</u> R defined on A if for elements $a, b, c \in A$ with $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

9. a relation is <u>an equivalence relation</u> if it is reflexive, symmetric and transitive

10. operations on relation are operations on sets (like union, intersection, difference, symmetric difference ($R \oplus S = (R - S) \cup (S - R)$))

11. composition of relations: for elements $a, b, c \in A$, and two relations R and S, if $(a, b) \in R$ and $(b, c) \in S$ then $(a, c) \in S \circ R$

12. composing a relation to itself: $R^n = R^{n-1} \circ R$ with the initial condition $R^1 = R$

13. a relation is transitive iff $R^n \subseteq R$

8.3 Representing Relations

representing relations using matrices

1. a zero-one matrix $M_R = [m_{ij}]$ can be used to represent a relation $R = \{(a_i, b_j)$ for some $i, j\}$ if we let $m_{ij} = 1$ iff $(a_i, b_j) \in R$.
2. if R is defined on a set, then M_R is a square matrix.
3. R is reflexive iff M_R has only 1s on its diagonal
4. R is irreflexive iff M_R has only 0s on its diagonal
5. R is symmetric iff $M_R = (M_R)^t$ (i.e. M_R is a symmetric matrix)
6. R is antisymmetric iff M_R has either $m_{i,j} = 0$ or $m_{j,i} = 0$ (or both) for all $i \neq j$
7. $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$ and $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$ and $M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1}$ and $M_{R^n} = M_R^{[n]}$

representing relations using digraphs

1. a digraph consists of a set V of vertices (nodes) and a set E of ordered pairs (arcs): if $a, b \in V$ and $(a, b) \in E$ then the arc (a, b) (or just (ab)) belongs to the digraph, where a is the initial vertex and b is the terminal vertex of the arc
2. the arc (a, a) is a loop
3. a relation can be modeled using a digraph where each arc of the digraph represents an element of the relation
4. a relation is reflexive iff there is a loop at every vertex
5. a relation is irreflexive iff there are no loops
6. a relation is symmetric iff every time the arc (a, b) is present, the arc (b, a) is present (if both arc (b, a) and (a, b) are presents, the two arcs may be replaced by one undirected edge)
7. a relation is antisymmetric iff for every pair of vertices a and b, at most one of the arcs (a, b) and (b, a) is present
8. a relation is transitive iff whenever the arcs (a, b) and (b, c) are present, then the arc (a, c) is present (make sure you use the loops in the case you have the arcs (a, b) and (b, a) since you'll need to have the loops (a, a) and (b, b))

8.5 Equivalence Relations

1. an equivalence relation R on A is a relation that is reflexive, symmetric and transitive

2. for an equivalence relation R, the equivalence class $[a]_R$ (or simply $[a]$) of an element $a \in A$ is the set containing a together with all elements related to a:

$$[a] = \{s : (a, s) \in R\} = \{s : (s, a) \in R\}$$

3. any element of the class can be a representative of the class, and so any element of the class can give the name of the class (the name of an equivalence class is not unique)

4. in particular, the equivalence classes for the equivalence relation "congruence modulo m" are called congruence classes modulo m
   Example: the classes of the equivalence relation congruence modulo 5 over the integers are
   $[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$
   $[1] = \{\ldots, -9, -4, 1, 6, 11, 16, 21, \ldots\}$
   $[2] = \{\ldots, -8, -3, 2, 7, 12, 17, 22, \ldots\}$
   $[3] = \{\ldots, -7, -2, 3, 8, 13, 18, 23, 28 \ldots\}$
   $[4] = \{\ldots, -6, -1, 4, 9, 14, 19, 24, 29, \ldots\}$

5. a partition of a set A is a collection of subsets A_i $(1 \leq i \leq t)$ of A such that

   - any two subsets are disjoint (i.e. $A_i \cap A_j = \emptyset, \forall i \neq j, 1 \leq i \neq j \leq t$)
   - no subset is empty (i.e. $A_i \neq \emptyset, \forall i, 1 \leq i \leq t$)
   - the union of the subsets is A itself $\left(\text{i.e. } A = \bigcup_{i=1}^{t} A_i\right)$

6. Example: The equivalence classes above: $[0], [1], [2], [3], [4]$, partition the integers into classes modulo 5. Note that this new partition can form a set with 5 whose elements can be added and multiplied, and the result is another element in the partition: $[3] + [4] = [2]$ since $(3 + 4)\,\underline{mod}5 = 7\,\underline{mod}5 = 2$

7. the equivalence classes of an equivalence relation R on A form a partition of the set A.

8. similarly, a partition of A induces an equivalence relation R whose equivalence classes are the partition sets (and thus the relation can be obtained)

2.3 Functions

1. a function f from A to B is an assignement of a unique value of B to each value of A. (note that this means each value of A can only be mapped to a unique value, and also, each value of A has to be mapped to some value of B.)

2. the set A above is called the domain, and the set B is called the codomain. A subset of the codomain makes the range of f, and that subset is the set of particular values of B that get assigned to values of A.

3. Let $a \in A$ and say that $f(a) = b$, of course with $b \in B$. Then b is called the image of a, and a is called the preimage of b. Then f is said to map a to b. The set of all values b will make up the range of f, as defined above.

4. two functions f and g are equal if they have the same domain and codomain, and $f(x) = g(x)$ for every value x of the domain

5. two functions can be added, subtracted, divided and multiply if they have the same domain (so that the new function will be defined)

6. a function is strictly increasing iff: $\forall x, y, \Big((x < y) \to (f(x) < f(y))\Big)$.

7. a function is increasing iff: $\forall x, y, \Big((x < y) \to (f(x) \leq f(y))\Big)$.

8. a function is strictly decreasing iff: $\forall x, y, \Big((x < y) \to (f(x) > f(y))\Big)$.

9. a function is decreasing iff: $\forall x, y, \Big((x < y) \to (f(x) \geq f(y))\Big)$.

10. a function is one-to-one or injective iff (that is if and only if):

    $$\forall x, y, \Big((f(x) = f(y)) \to (x = y)\Big)$$

    For example, the function $f(x) = 2x + 3, f : \mathbb{R} \to \mathbb{R}$ is one-to-one, but the function $f(x) = 2x^2 + 3, f : \mathbb{R} \to \mathbb{R}$ is not (prove them to convince yourself).

11. a function is onto or surjective iff:

    $$\forall y \in B, \exists x \in A \; (f(x) = y)$$

    For example, the function $f(x) = 2x + 3, f : \mathbb{R} \to \mathbb{R}$ is onto, but the function $f(x) = 2x^2 + 3, f : \mathbb{R} \to \mathbb{R}$ is not (prove them to convince yourself).

12. a function that is both one-to-one and onto is a one-to-one correspondence or bijective. All linear functions are bijectives from reals to the reals ($f : \mathbb{R} \to \mathbb{R}$)

13. if a function $f : A \to B$ is bijective (or one-to-one correspondence) with $f(x) = y$ then there is an inverse function $f^{-1} : B \to A$ with $f(y) = x$. For example, for $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$, the inverse function is $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = \frac{x-1}{2}$ (note that the inverse function is not defined by $f(x) = \frac{1}{2x+1}$)

14. a one-to-one correspondence f is called invertible because there is a function (namely f^{-1}) that is the inverse of f

15. the composition of two functions f and g is defined by $(f \circ g)(x) = f(g(x))$. For example, for $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 2$ and $g(x) = x^2 - 3$, then $(f \circ g) : \mathbb{R} \to \mathbb{R}$, $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 3(x^2 - 3) - 2 = 3x^2 - 11$, however, $(g \circ f) : \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) = g(f(x)) = g(3x - 2) = (3x - 2)^2 - 3 = 9x^2 - 12x + 1$

16. note that $f \circ f^{-1} = f^{-1} \circ f = id$, where id is the identity function that takes any value to itself ($id(x) = x$ for any domain and codomain). And so $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x values of the domain

17. the inverse image of a set $\{y\}$ in the Range is $f^{-1}(\{y\}) = a$ iff $f(a) = y$ (so it is the value that got mapped to y). For the inverse image of a set, the function does not have to be bijective, and so the inverse image could be more than one value. Example: let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $f^{-1}(4) = \{-2, 2\}$ and $f^{-1}(\{4, 5\}) = \{-\sqrt{5}, -2, 2, \sqrt{5}\}$. Note that the inverse image is defined for a set, not for a value (and so the set could have just one element, as shown above with $S = \{4\}$)–definition on page 147

18. the graph of the function is the set of ordered pairs $(x, f(x))$ for all x values in the domain

19. the floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{R}$ is the largest integer that is less than or equal to x (Example $\lfloor 3.87 \rfloor = 3$ and $\lfloor -3.87 \rfloor = -4$

20. the ceiling function $\lceil x \rceil : \mathbb{R} \to \mathbb{R}$ is the smallest integer that is greater than or equal to x (Example $\lceil 3.27 \rceil = 4$ and $\lfloor -3.87 \rfloor = -3$

21. properties of floor and ceiling functions page 144 (note that n is an integer, but x can be any real number)

22. the factorial function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$. For example $f(4) = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$